Semi-automatic proof
of
Strong connectivity

jean-jacques.levy@inria.fr

journées PPS, 12-10-2017

Plan

- motivation
- algorithm
- formal proof
- other systems
- conclusion

.. joint work (in progress) with **Ran Chen** [VSTTE 2017])

also cooperation with Cyril Cohen, Laurent Théry, Stephan Merz

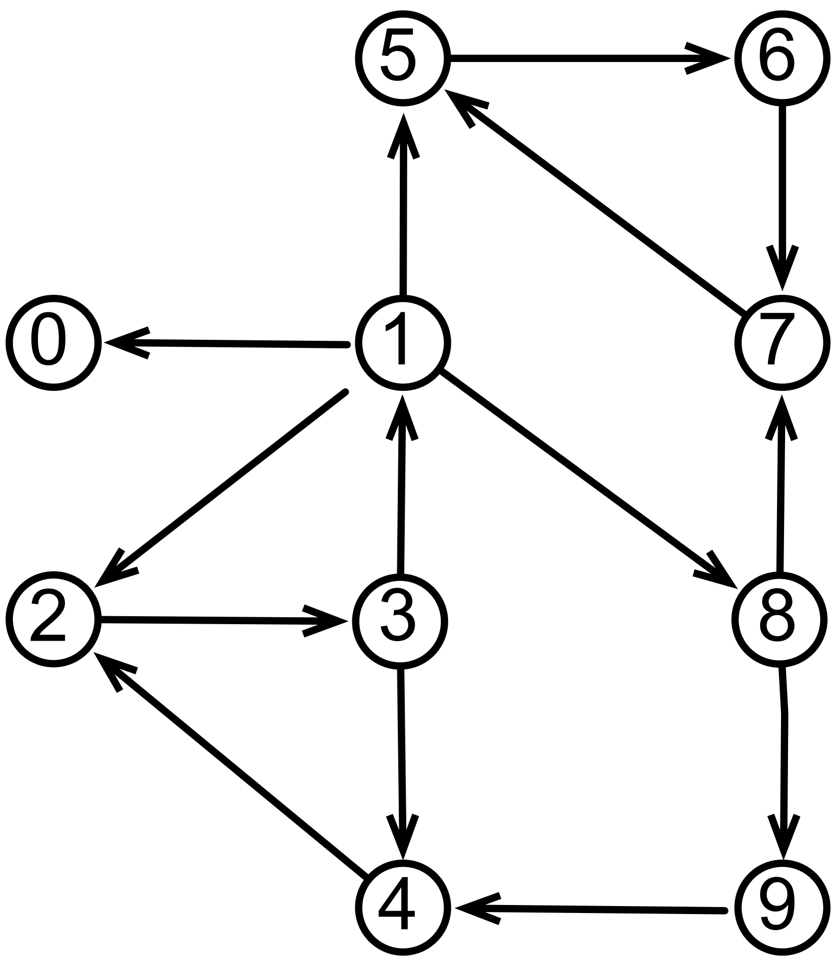
Motivation

- nice algorithms → **simple** formal proofs
- **fully** published in articles or journals
- how to publish formal proofs ?
- formal proofs should be **exact** and **readable** (by human)
- mix automatic and interactive proofs
- first-order logic is **easy** to understand, but **not** expressive
- algorithms on graphs = a good testbed

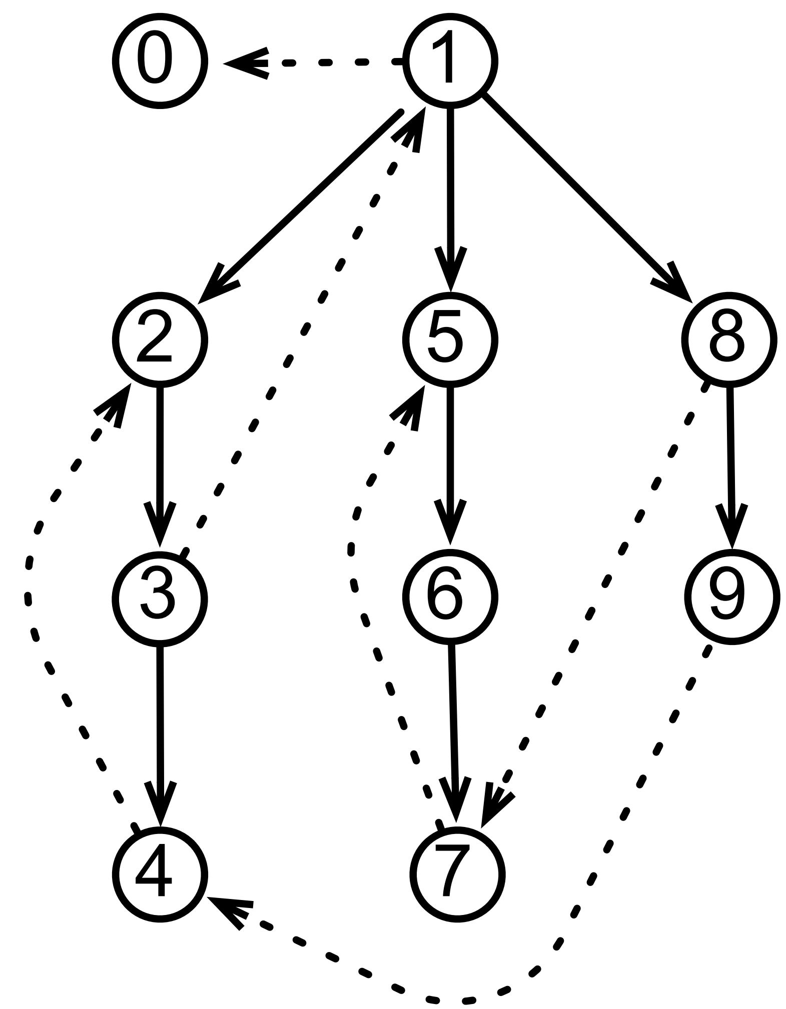
One-pass linear-time algorithm

[tarjan 1972]

Depth-first-search

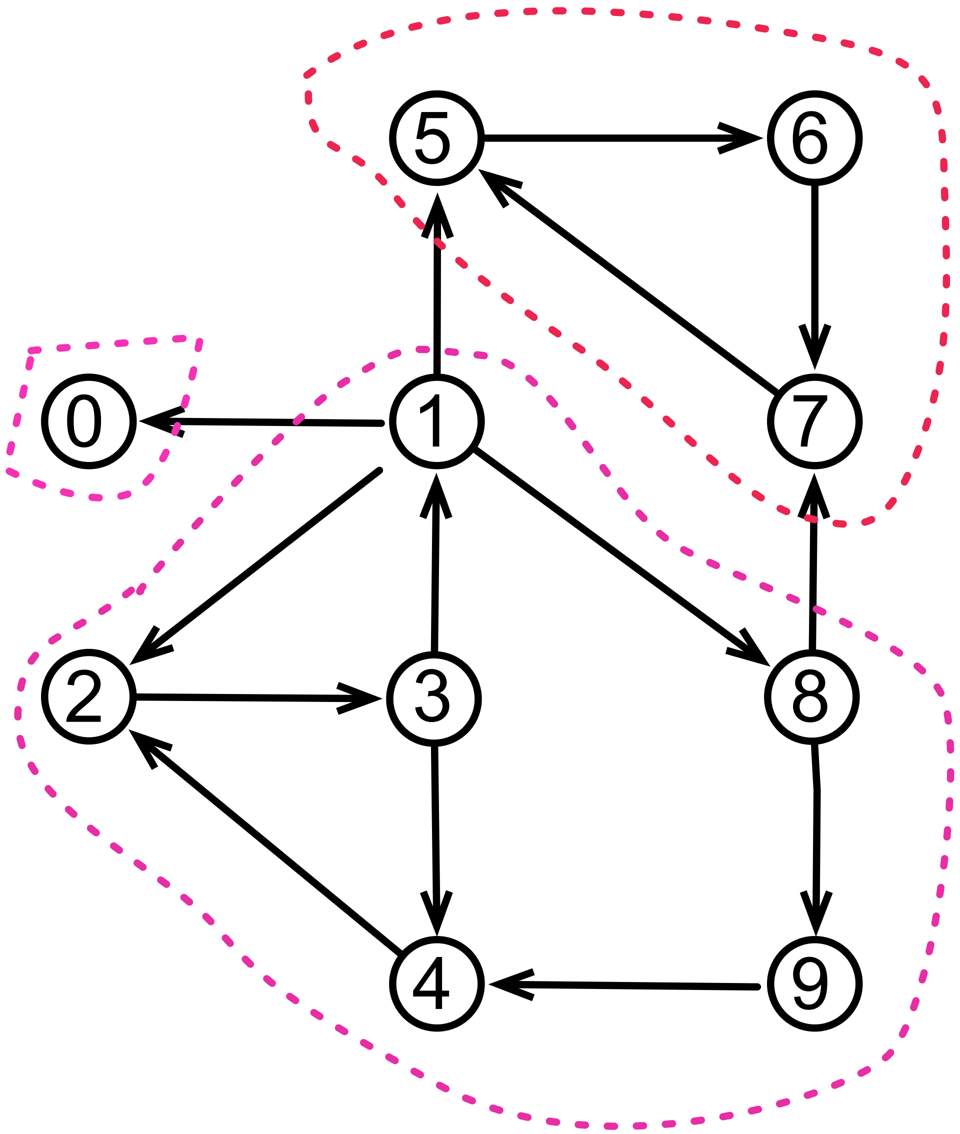


graph

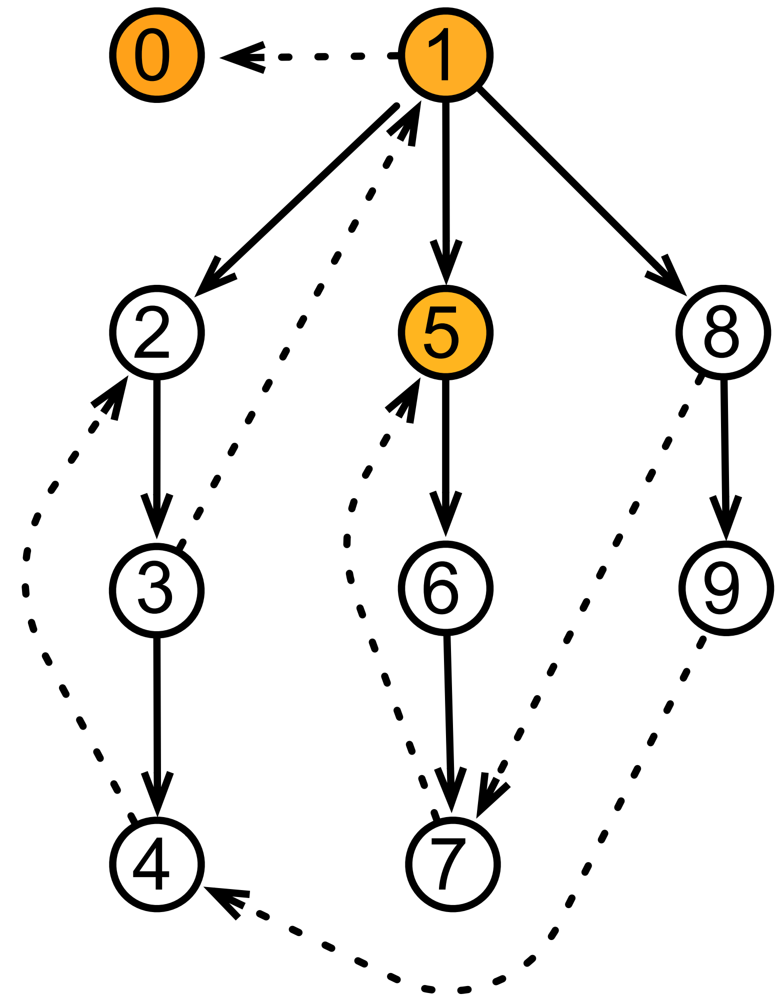


spanning tree (forest)

The algorithm (1/3)

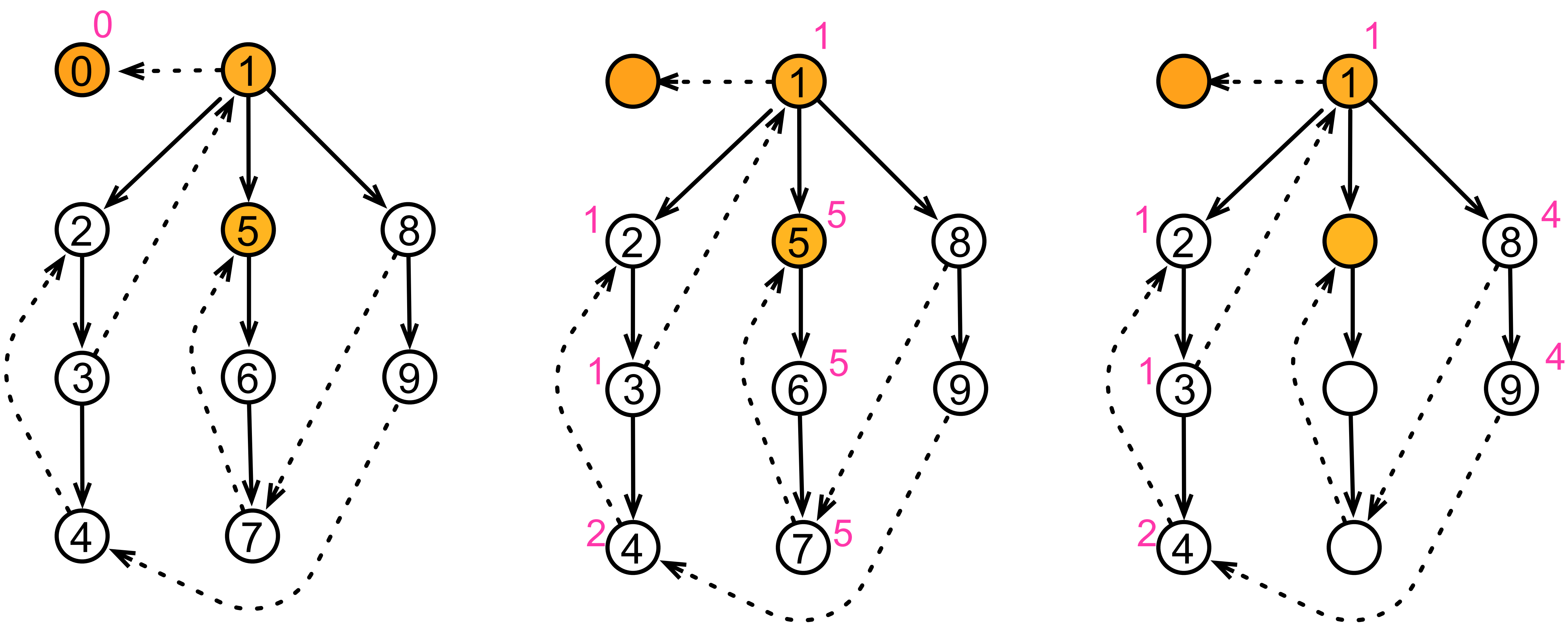


3 SCCs (strongly connected components)



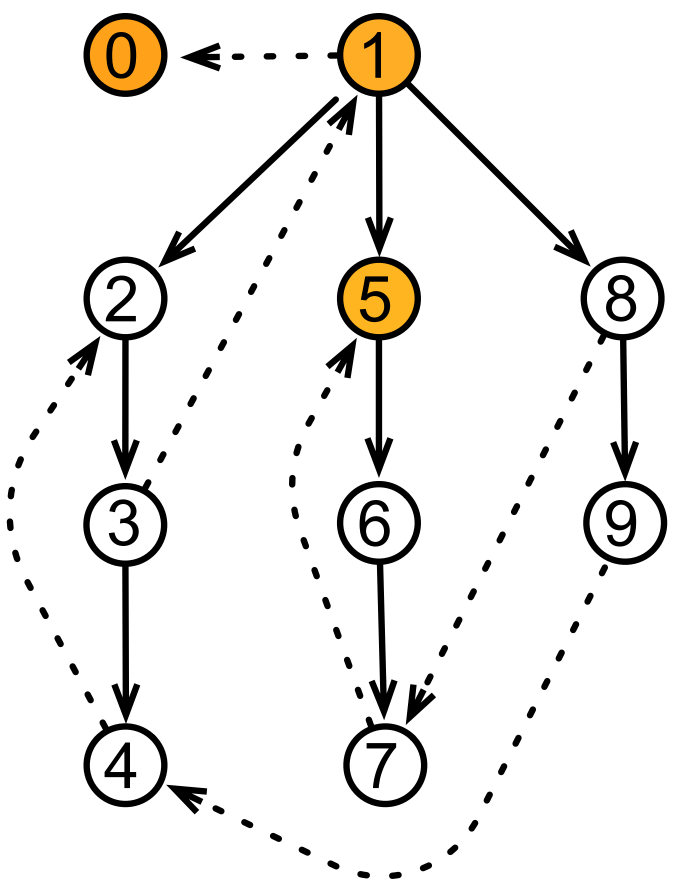
3 vertices are their bases

The algorithm (2/3)



$$\mathit{LOWLINK}(x) = \min\left(\ \{\mathit{num}[x]\}\ \cup\ \{\mathit{num}[y]\ \mid\ x \xrightarrow{*} \dashrightarrow y \ \wedge x \text{ and } y \text{ are in same connected component}\}\ \right)$$

The algorithm (3/3)



successive values of the working stack

										increasing rank
0	1	1	1	1	1	1	1	1	1	0
		2	2	2	2	2	2	2	2	1
			3	3	3	3	3	3	3	2
				4	4	4	4	4	4	3
					5	5	5	8	8	4
						6	6		9	5
							7			6

The program

```
let rec printSCC (x: int) (s: stack int)
    (num: array int) (sn: ref int) =
  Stack.push x s;
  num[x] ← !sn; sn := !sn + 1;
  let low = ref num[x] in
  foreach y in (successors x) do
    let m = if num[y] = -1
      then printSCC y s num sn
      else num[y] in
    low := Math.min m !low
  done;
```

```
if !low = num[x] then begin
  repeat
    let y = Stack.pop s in
    Printf.printf "%d " y;
    num[y] ← max_int;
    if y = x then break;
  done;
  Printf.printf "\n";
  low := max_int;
end;
return !low;
```

- print each component on a line

Imperative style

Proof in algorithms books (1/2)

- consider the spanning trees (forest)
- tree structure of strongly connected components
- 2-3 lemmas about ancestors in spanning trees

LEMMA 10. *Let v and w be vertices in G which lie in the same strongly connected component. Let F be a spanning forest of G generated by repeated depth-first search. Then v and w have a common ancestor in F. Further, if u is the highest numbered common ancestor of v and w, then u lies in the same strongly connected component as v and w.*

$$\begin{aligned} \mathit{LOWLINK}(x) = \min\,(\ \{\mathit{num}[x]\}\ \cup\ \{\mathit{num}[y]\ \mid\ x \stackrel{*}{\Longrightarrow}\hookrightarrow y \\ \wedge\ x \text{ and } y \text{ are in same} \\ \text{connected component}\}\ ) \end{aligned}$$

LEMMA 12. *Let G be a directed graph with* LOWLINK *defined as above relative to some spanning forest F of G generated by depth-first search. Then v is the root of some strongly connected component of G if and only if* LOWLINK $(v) = v$.

Proof in algorithms book (2/2)

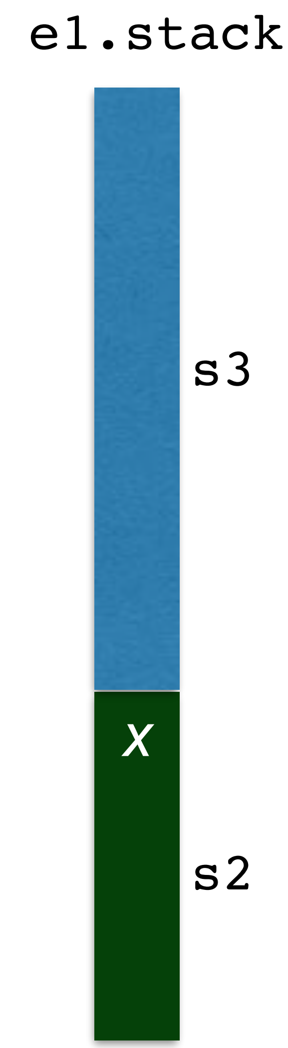
- give the program
- proof ⟷ program
- that part of the proof is very informal

Our program (1/3)

```
let rec dfs1 x e =
  let n = e.sn in
  let (n1, e1) = dfs (successors x) (add_stack_incr x e) in
  let (s2, s3) = split x e1.stack in
  if n1 < n then (n1, e1) else
    (max_int(), {stack = s3; sccs = add (elements s2) e1.sccs;
      sn = e1.sn; num = set_max_int s2 e1.num})

with  dfs roots e = if is_empty roots then (max_int(), e) else
  let x = choose roots in
  let roots' = remove x roots in
  let (n1, e1) = if e.num[x] ≠ -1 then (e.num[x], e) else dfs1 x e in
  let (n2, e2) = dfs roots' e1 in (min n1 n2, e2)

let tarjan () =
  let e0 = {stack = Nil; sccs = empty; sn = 0; num = const (-1)} in
  let (_, e') = dfs vertices e0 in e'.sccs
```



returns *LOWLINK*(x) and new environment

Functional programming

Formal proof

using Why3

Plan of proof (1/2)

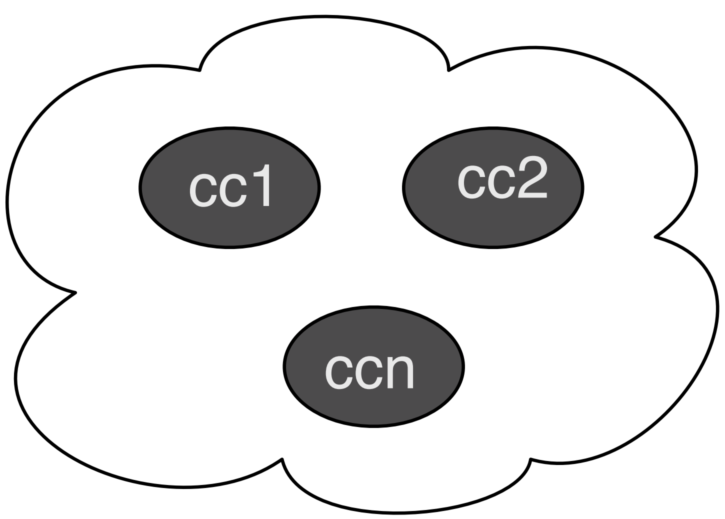
- define **reachability** in graphs and SCCs
- prove a few lemmas about positions in stacks (**ranks**)
- define **invariants** on environments
- give **pre-post conditions** for functions
- add a few intermediate **assertions** in function bodies

- avoid paths, prefer edges

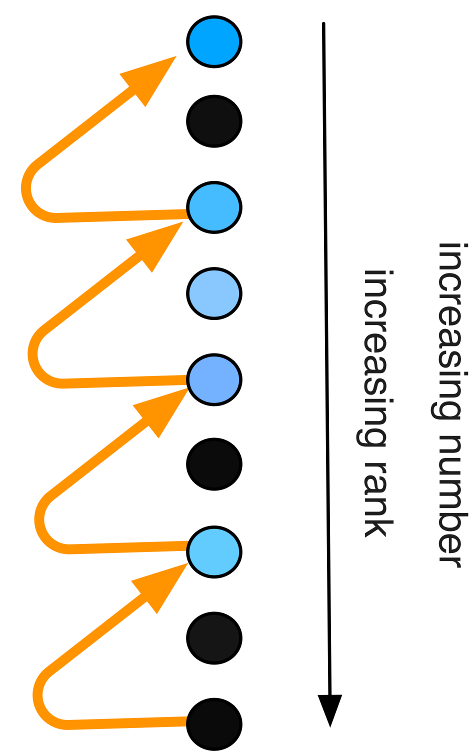
Plan of proof (2/2)

- vertices have colors
  - white = unvisited - gray = being visited - black = visited
- invariant on environment

sccs



stack



vertex in stack reaches all vertices with higher rank

Invariants

```
type env = {ghost blacks: set vertex; ghost grays: set vertex;
          stack: list vertex; sccs: set (set vertex);
          sn: int; num: map vertex int}
```

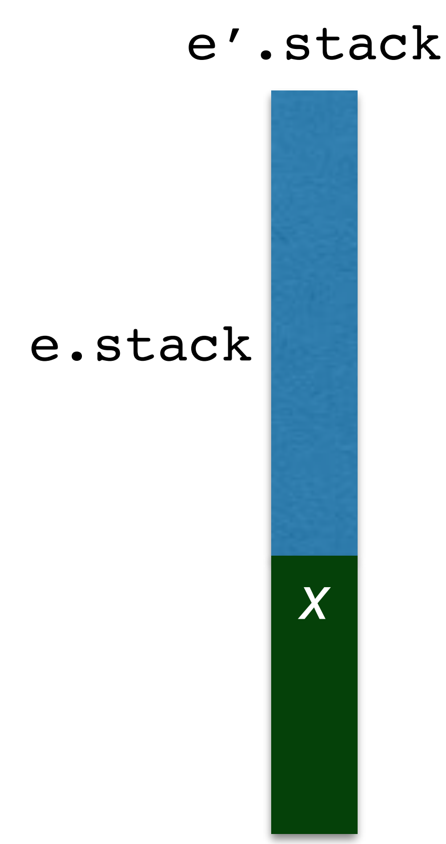
Pre/Post-conditions

```
let rec dfs1 x e  =
requires {mem x vertices} (* R1 *)
requires {access_to e.grays x} (* R2 *)
requires {not mem x (union e.blacks e.grays)} (* R3 *)
```

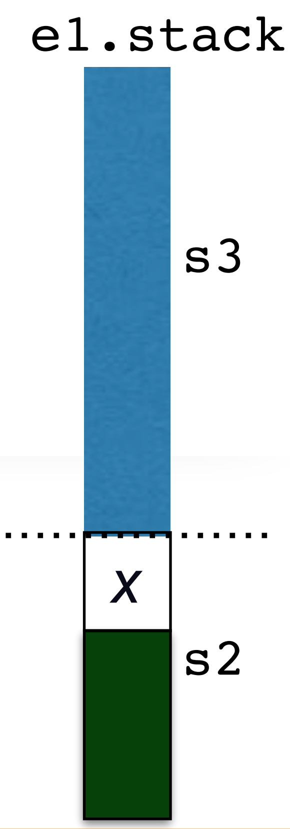
$$\texttt{e.sccs} \subseteq \texttt{e'.sccs}$$

$$\texttt{e.blacks} \subseteq \texttt{e'.blacks}$$

$$\texttt{e.grays} = \texttt{e'.grays}$$



Assertions




```
let n = e.sn in
let (n1, e1) =
    dfs' (successors x) (add_stack_incr x e) in
let (s2, s3) = split x e1.stack in


if n1 < n then begin

  (n1, add_blacks x e1) end
else  begin


  (max_int(), {blacks = add x e1.blacks; grays = e.grays;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_max_int s2 e1.num}) end
```

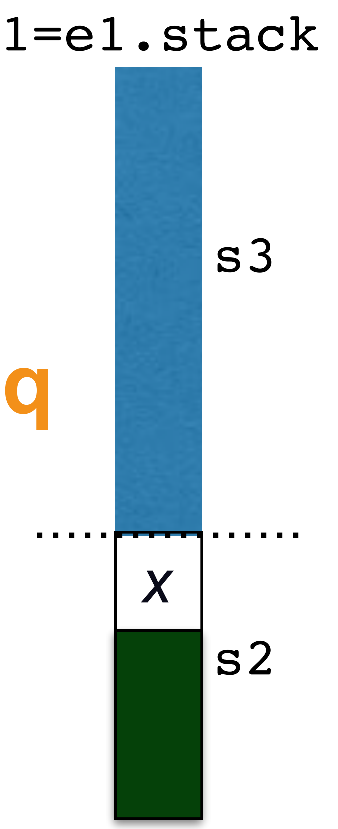
Coq

[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Assertions

```
assert {forall y. in_same_scc y x -> lmem y s2};
```

Coq



- proof by contradiction: $\exists y,\ \ \text{in\_same\_scc}\ y\ x\ \wedge\ y \notin s2$
- $\exists x' y',\ \ \text{reachable}\ x\ x'\ \wedge\ \text{edge}\ x'\ y'\ \wedge\ \text{reachable}\ y'\ y\ \wedge\ x' \in s2\ \wedge\ y' \notin s2$
- 3 cases:

[1] y' is white

$x' = x$ then $y' \in \text{successors}\ x \rightarrow y'$ is black

$x' \neq x$ then x' is black $\rightarrow \neg\ \text{no\_black\_to\_white}\ \ b1\ g1$

[2] $y' \in$ e1.sccs then $\text{in\_same\_scc}\ \ y'\ x \rightarrow x$ is black

[3] $y' \in s3 \rightarrow \text{rank}\ y'\ s1\ <\ \text{rank}\ x\ s1 \rightarrow \text{e1.num}[y']\ <\ \text{e1.num}[x] = \text{e.num}[x] = n$

$x' = x$ then $y' \in \text{successors}\ x \longrightarrow n1\ \leq\ \text{e1.num}[y']$

$x' \neq x$ then $\text{xedge\_to}\ \ s1\ (\text{Cons}\ \ x\ s3)\ \ y'$

Proof stats

provers	Alt-Ergo	CVC3	CVC4	Coq	E-prover	Spass	Yices	Z3	all	#VC	#PO
38 lemmas	2.35	0.23	5.79		0.66	0.75	0.21		9.99	77	38
split	0.09	0.2							0.29	6	6
add_stack_incr	0.01								0.01	1	1
add_blacks	0.01								0.01	1	1
set_max_int	0.02								0.02	1	1
dfs1	53.52	12.88	36.39	3.06	28.06			9.01	142.92	218	24
dfs	4.6	0.23	11.63					0.31	16.77	51	35
tarjan	0.44								0.44	16	6
total	61.04	13.54	53.81	3.06	28.72	0.75	0.21	9.32	170.45	371	112

[http://jeanjacqueslevy.net/why3/graph/abs/scct/1-7/scc.html]

Other systems

Coq / ssreflect

[cyril cohen, laurent théry, JJL]

- port in 1 week
- graphs and finite sets already in mathematical components
- problems with termination (hacky & higher-order)
- 920 lines

[http://github.com/CohenCyril/tarjan]

Isabelle / HOL

[stephan merz]

- port in 1 month
- use many strategies (metis, blast, sledgehammer)
- still problems with proving termination
- 31 pages

[http://jeanjacqueslevy.net/why3/graph/abs/scct/isa/Tarjan.pdf]

F*

`[kenji maillard, catalin hritcu]`

- start discuss with them
- Z3 single automatic prover
- ??

Conclusion

Future work

- library for formal proofs on graphs
- other graph algorithms
- **beyond** graphs …
- teaching formal methods on **test cases**
- **imperative** programs

[http://jeanjacqueslevy.net/why3]