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AN ALGEBRAIC INTERPRETATION OF THE $\lambda\beta K$-CALCULUS; AND AN APPLICATION OF A LABELLED λ-CALCULUS

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1. Introduction

A wide range of λ-calculus models has been proposed by Scott [11, 12]. In these interpretations, the interconvertibility relation among λ-expressions is extended by mainly equating the unsolvable terms (i.e., expressions M such that for any arguments $N_1, N_2, \cdots, N_k$ the expression $MN_1 N_2 \cdots N_k$ has no normal form). This extension has been shown by Barendregt [1] and Wadsworth [14] to be consistent. Hyland [5] and Wadsworth [14] showed the adequacy of most of Scott's models from a computational point of view; more precisely, each expression is equal to the limit of its approximations in these models. We will try to go in the reverse direction in the first part of this paper and define the value of an expression from its set of approximations. Then we prove that, as usual, our interpretation defines a congruence on the language of λ-expressions (see Milner [8]). For this we follow Welch [15] who made a conjecture about the completeness, in the reducibility sense, of "inside-out reductions".* This conjecture is proved in the second part of this paper by introducing a "labelled λ-calculus", which the author believes to be a useful tool for some λ-calculus problems. The results in this paper are related to the ones in Hyland [4] and Welch [16]. The definition of our interpretation is very similar to that of Nivat [10] and Vuillemin [13] used for systems of recursively defined functions. Most results appeared in the author's thesis [6].

2. Syntax

We consider the set Λ of λ-expressions, built from an infinite alphabet V of variables, which is the minimal set containing

(1) x (variable),
(2) (λxM) (abstraction),
(3) (MN) (application),

where x is in V and M, N are already in Λ. We will use the standard abbreviations where

* In fact, there is a long proof by Welch. We just show an alternative proof which is mavbe simpler.

$$MNN_1N_2\cdots N_k \text{ stands for } (\cdots(((MN)N_1)N_2)\cdots N_k),$$
$$(\lambda x_1x_2\cdots x_m\cdot M) \text{ stands for } (\lambda x_1(\lambda x_2\cdots(\lambda x_mM)\cdots)),$$

M, N, N_i being expressions in Λ, and x_i being variables. We shall also omit the outer-most parentheses of an expression. The usual notions of free and bound variables are assumed defined and we denote by $M[x\backslash N]$ the substitution of N for the free occurences of x in M.

We consider only two rules of conversion: the α and β rules. If N derives from M by an α-conversion, we write $M\underset{\alpha}{\rightarrow}N$. Similarly, we have $M\underset{\beta}{\rightarrow}N$, and a reduction (possibly of length zero) using only α-conversion from M to N is written $M\overset{*}{\underset{\alpha}{\rightarrow}}N$. Hence $M\overset{*}{\underset{\beta}{\rightarrow}}N$ and $M\overset{*}{\underset{\alpha,\beta}{\longrightarrow}}N$ mean that M reduces by a sequence of β-reductions, or α-conversions and β-reductions, to N. We often forget α-conversions and $M\rightarrow N$ or $M\overset{*}{\rightarrow}N$ are understood as $M\underset{\beta}{\rightarrow}N$ or $M\overset{*}{\underset{\alpha,\beta}{\longrightarrow}}N$. Equality must also be considered as equality modulo some α-conversions. We will try to use the usual terminology (residuals, standard reductions, etc.) defined in [2, 3]. We also make use of the context notation (see [9, 14]).

Let us first remark that Λ can also be considered as the smallest set containing

(i) $\lambda x\cdot M$ (abstraction),

(ii) $xM_1M_2\cdots M_n$ (head normal form),

(iii) $(\lambda x\cdot M)NM_1M_2\cdots M_n$,

if x is a variable and M, N, M_i are expressions of Λ. More generally, a head normal form is any expression of the form $\lambda x_1x_2\cdots x_m\cdot xM_1M_2\cdots M_n$ where $m, n\geqslant 0$ (see [14]). Other expressions are of the form $\lambda x_1x_2\cdots x_m\cdot(\lambda x\cdot M)NM_1M_2\cdots M_n$ and have a head redex $(\lambda x\cdot M)N$. If $M\overset{*}{\rightarrow}N$ and N is an abstraction (respectively a head normal form) we say that M has an abstraction form (respectively a head normal form).

Proposition 2.1. *If M has an abstraction form, then M has a minimal abstraction form $\lambda x\cdot N_0$, i.e., we have $M\overset{*}{\rightarrow}\lambda x\cdot N_0$, and for any $\lambda x\cdot N$ such that $M\overset{*}{\rightarrow}\lambda x\cdot N$ we have $\lambda x\cdot N_0\overset{*}{\rightarrow}\lambda x\cdot N$.*

Proof. M can only be of form (i) or (iii). In the first case, we have $M=\lambda x\cdot N_0$. Otherwise, for any $\lambda x\cdot N$ such that $M\overset{*}{\rightarrow}\lambda x\cdot N$, by the standardization theorem there is a standard reduction

$$M=M_0\overset{R_1}{\longrightarrow}M_1\overset{R_2}{\longrightarrow}M_2\overset{R_3}{\longrightarrow}\cdots\overset{R_n}{\longrightarrow}M_n=\lambda x\cdot N$$

from M to $\lambda x\cdot N$. Let M_k be the first M_i which is an abstraction. Then, since the reduction is standard, the redexes R_j contracted between M_{j-1} and M_j are the head redexes of M_{j-1} for $1\leqslant j\leqslant k$. So each standard reduction from M to some $\lambda x\cdot N$ has a common initial part

$$M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \xrightarrow{R_3} \cdots \xrightarrow{R_k} M_k. \quad \square$$

Proposition 2.2. *If M has a head normal form, then M has a minimal one.*

The proof is very similar to the preceding one. In both cases, the minimal form is obtained by contracting head redexes until an expression of the desired form is reached.

3. Approximations

We still follow Wadsworth [14] and define the direct approximation $\phi(M)$ of an expression M by

$$\phi(\lambda x \cdot M) = \lambda x \cdot \phi(M),$$
$$\phi(xM_1M_2 \cdots M_n) = x(\phi(M_1))(\phi(M_2)) \cdots (\phi(M_n)),$$
$$\phi((\lambda x \cdot M)NM_1M_2 \cdots M_n) = \Omega,$$

where Ω is an extra constant. Basically, $\phi(M)$ is obtained from M by replacing all (outermost) redexes of M by Ω and substituting Ω for ΩM until normal form is obtained. If Ω is understood to be "undefined", $\phi(M)$ is the information we have from M without contracting its redexes. There is a slight modification of Wadsworth's definition, because we do not want to identify Ω and $\lambda x \cdot \Omega$.

We define $\mathcal{N}$ by $\mathcal{N} = \phi(\Lambda)$. Obviously, $\mathcal{N}$ is the set of expressions in ω-β normal forms. More precisely, $\mathcal{N}$ is the minimal set containing:

$$\begin{cases} \Omega \\ \lambda x \cdot a \\ x a_1 a_2 \cdots a_n \end{cases}$$

if x is a variable and a, a_i are already in $\mathcal{N}$. By considering Ω as a minimal element in $\mathcal{N}$, and extending monotonically, we get the following partial order $<$ in $\mathcal{N}$:

$$\Omega < a,$$
$$\lambda x \cdot a < \lambda x \cdot b \quad \text{if } a < b,$$
$$x a_1 a_2 \cdots a_n < x b_1 b_2 \cdots b_n \quad \text{if } a_i < b_i \text{ for } 1 \leqslant i \leqslant n,$$

where a, b, a_i are expressions of $\mathcal{N}$, x is a variable, and $n \geqslant 0$. Here, we must take care of α-conversion and the order $<$ is, in fact, an order between equivalence classes defined on $\mathcal{N}$ by the α-interconvertibility. So if $a \underset{\alpha}{\overset{*}{\rightarrow}} a'$ and $b \underset{\alpha}{\overset{*}{\rightarrow}} b'$, we have $a < b$ iff $a' < b'$. Moreover, we notice that $a < b$ iff there are M, N in Λ such that $\phi(M) = a$, $\phi(N) = b$ and $M \overset{*}{\rightarrow} N$.

Proposition 3.1. *The set $\mathcal{N}$ is a semi-lattice where every directed subset is a lattice.* More precisely,*

(1) *$\mathcal{N}$ has a minimal element Ω;*

(2) *for any pair, a, b of elements in $\mathcal{N}$, there exists a greatest lower bound $a \cap b$ (meet operation);*

(3) *for any pair a, b of elements in $\mathcal{N}$ which are dominated by a common upper bound, there exists a least upper bound $a \cup b$ (join operation).*

The proof is trivial, and obviously we can give the inductive definitions of $a \cup b$ and $a \cap b$ (up to some α-conversions, as for the definition of $<$ above):

$$(\lambda x \cdot a) \cap (\lambda x \cdot b) = \lambda x \cdot (a \cap b),$$

$$(x a_1 a_2 \cdots a_n) \cap (x b_1 b_2 \cdots b_n) = x(a_1 \cap b_1)(a_2 \cap b_2) \cdots (a_n \cap b_n),$$

$$a \cap b = \Omega \text{ otherwise,}$$

and

$$\Omega \cup a = a \cup \Omega = a,$$

$$(\lambda x \cdot a) \cup (\lambda x \cdot b) = \lambda x \cdot (a \cup b),$$

$$(x a_1 a_2 \cdots a_n) \cup (x b_1 b_2 \cdots b_n) = x(a_1 \cup b_1)(a_2 \cup b_2) \cdots (a_n \cup b_n),$$

$$a \cup b \text{ is not defined otherwise,}$$

where x is a variable, a, b, a_i, b_i are expressions of $\mathcal{N}$, and $n \geqslant 0$. The set $\mathcal{N}$ is also complete for the $\cap$ operation, i.e., each subset X of $\mathcal{N}$ has a greatest lower bound $\cap X$ in $\mathcal{N}$. Moreover, the order $<$ is well-founded in $\mathcal{N}$ and we have no infinite strictly decreasing chains in N.

To any expression M, Wadsworth [14] associates a set of approximations $\mathcal{A}(M)$, which is the set of direct approximations of all expressions reducible from M:

$$\mathcal{A}(M) = \{\phi(N) \colon M \xrightarrow{*} N\}.$$

We briefly review some descriptive properties.

Proposition 3.2. *The set $\mathcal{A}(M)$ of approximations of any λ-expression M is a sub-lattice of $\mathcal{N}$ (with the same meet and join operations as in $\mathcal{N}$).*

Proof. We need only to show that $\cap$ and $\cup$ are closed in $\mathcal{A}(M)$. Suppose a, b are in $\mathcal{A}(M)$.

(1) $a \cap b$ is in $\mathcal{A}(M)$ by induction on the size $\| a \cap b \|$ of $a \cap b$. There are three cases.

* If $(D, <)$ is a partial order structure, a directed subset X of D is such that for any a, b in X, there is a c in X such that $a < c$, $b < c$ (a notion similar to that of ascending chains if D is denumerable). See Scott [11].

(a) If $a \cap b = \Omega$, then $a \neq \Omega$ and $b \neq \Omega$ is impossible because of proposition 3.1. Therefore $a = \Omega$ or $b = \Omega$ which implies $\phi(M) = \Omega$, and then $a \cap b$ is in $\mathcal{A}(M)$.

(b) If $a \cap b = \lambda x \cdot c_1$, then $a = \lambda x \cdot a_1$ and $b = \lambda x \cdot b_1$. Hence, M has an abstraction form and, by Proposition 2.1, a minimal one $\lambda x \cdot M_0$. As a_1 and b_1 are in $\mathcal{A}(M_0)$, we know by induction that $a_1 \cap b_1$ is in $\mathcal{A}(M_0)$. Thus, $\lambda x \cdot (a_1 \cap b_1)$ is in $\mathcal{A}(\lambda x \cdot M_0)$, and $a \cap b$ is in $\mathcal{A}(M)$.

(c) If $a \cap b = xc_1c_2 \cdots c_n$, we have the same proof using Proposition 2.2.

(2) The Church–Rosser Theorem shows the existence of a c such that $a < c$ and $b < c$. Hence, $a \cup b$ is defined and a similar proof, based on an induction on the size $\| a \cup b \|$, shows that $a \cup b$ is in $\mathcal{A}(M)$. □

Proposition 3.3. *For any a in $\mathcal{A}(M)$ there is a minimal λ-expression N_a such that:*

(i) $M \xrightarrow{*} N_a$;

(ii) $\phi(N_a) = a$;

(iii) *for any N such that $M \xrightarrow{*} N$ and $a < \phi(N)$, $N_a \xrightarrow{*} N$.*

Proof. Follows by induction on the size of a. There are three cases.

(1) $a = \Omega$. Then $\phi(M) = \Omega = a$ and $N_a = M$.

(2) $a = \lambda x \cdot a_1$. Then if $M \xrightarrow{*} N$ and $a < \phi(N)$, we have $N = \lambda x \cdot N_1$, $\phi(N) = \lambda x \cdot \phi(N_1)$, and $a_1 < \phi(N_1)$. By Proposition 2.1, there is a minimum abstraction form $\lambda x \cdot M_0$ of M. Hence $M_0 \xrightarrow{*} N_1$ and by induction there is an N_{a_1} minimal for a_1 and reducible from M_0. Hence, if $N_a = \lambda x \cdot N_{a_1}$, we have

$$M \xrightarrow{*} \lambda x \cdot M_0 \xrightarrow{*} N_a = \lambda x \cdot N_{a_1} \xrightarrow{*} \lambda x \cdot N_1 = N,$$

and $\phi(N_a) = \lambda x \cdot \phi(N_{a_1}) = \lambda x \cdot a_1 = a$.

(3) $a = xa_1a_2 \cdots a_n$. The proof is as in (2), but we now need Proposition 2.2. □

Hence, generalizing Propositions 2.1 and 2.2, we have a minimal expression N_a for any approximation a. We reach it by head reductions (leftmost outer-most reductions) until a head normal form (if necessary) is reached and repeating this process on arguments of the head normal form (when necessary). We notice also that if a, b are in some $\mathcal{A}(M)$, then $a < b$ iff for any N' such that $M \xrightarrow{*} N'$ and $\phi(N') = b$, there is an N such that $M \xrightarrow{*} N$, $\phi(N) = a$ and $N \xrightarrow{*} N'$.

4. The interpretation domain

In the set Λ, some λ-expressions have a finite set of approximations; we call them expressions of finite information. The other expressions are of infinite information and, in order to be able to speak of their value, we will complete the set $\mathcal{N}$ of ω-β normal forms by adding infinite points. Let $\bar{\mathcal{N}}$ be the set of all directed subsets of $\mathcal{N}$:

$$\bar{\mathcal{N}} = \{S\colon\ S \text{ directed},\ S \subset \mathcal{N}\}.$$

We can extend the relation $<$ to $\bar{\mathcal{N}}$ by defining for S and S' in $\bar{\mathcal{N}}$

$$S \subseteq S' \text{ iff } \forall a \in S,\ \exists b \in S'.\ \text{s.t. } a < b$$

In order to keep an ordering, we define a quotient set

$$\hat{\mathcal{N}} = \bar{\mathcal{N}} / \equiv,$$

where

$$S \equiv S' \text{ iff } S \subseteq S' \subseteq S.$$

Hence, if we denote by $[S]$ the equivalence class of S in $\hat{\mathcal{N}}$, we have in $\hat{\mathcal{N}}$

$$[S] \subseteq [S'] \text{ iff } S \subseteq S'.$$

Proposition 4.1. *The set $\hat{\mathcal{N}}$ is a semi-lattice where every directed subset is a lattice. More precisely,*

(1) *$\mathcal{N}$ has a minimal element* $[\{\Omega\}]$,

(2) *any pair of elements* $[S]$, $[S']$ *in $\hat{\mathcal{N}}$ has a greatest lower bound* $[S] \cap [S']$,

(3) *any pair of elements* $[S]$, $[S']$ *in $\hat{\mathcal{N}}$ which is dominated by a common upper bound has a least upper bound* $[S] \cap [S']$.

The proof is obvious and the definitions of $\cap$ and $\cup$ in $\hat{\mathcal{N}}$ are given by

$$[S] \cap [S'] = [\{a \cap b \mid a \in S, b \in S'\}],$$

$$[S] \cup [S'] = [\{a \cup b \mid a \in S, b \in S'\}].$$

But $\hat{\mathcal{N}}$ has a richer structure. Using Scott's terminology (see for instance [11]), we have:

Proposition 4.2. *The domain $\hat{\mathcal{N}}$ is*

(1) *complete for directed subsets of $\hat{\mathcal{N}}$,*

(2) *algebraic, since $\hat{\mathcal{N}}$ admits a denumerable basis of isolated elements* $[\{a\}]$ *where $a \in \mathcal{N}$.*

This means that every directed subset X of $\hat{\mathcal{N}}$ has a least upper bound $\bigcup X$ and that each element of $\hat{\mathcal{N}}$ is the least upper bound of the finite information points $[\{a\}]$ (where $a \in \mathcal{N}$) which are below it. The proof follows from the construction of $\hat{\mathcal{N}}$ and we skip it. The method we use for the completion of $\hat{\mathcal{N}}$ is equivalent to the one of Vuillemin [13].

5. Interpretation of λ-expressions

We associate with any expression M an element $\hat{\mathcal{N}}$ by the following equation

$$\mathcal{J}(M) = [\mathcal{A}(M)],$$

and we can thus induce a partial preorder on Λ defined by [$M \subseteq M'$ iff $\mathcal{J}(M) \subseteq \mathcal{J}(M')$.] By the definition of $\hat{\mathcal{N}}$, we have

$$M \subseteq M' \text{ iff } \forall N \text{ s.t. } M \xrightarrow{*} N, \exists N' \text{ s.t. } M' \xrightarrow{*} N' \text{ and } \phi(N) < \phi(N').$$

We write $M \equiv M'$ for $M \subseteq M' \subseteq M$ and we expect the usual properties for our interpretation $\mathcal{J}$. Moreover, if X is a directed subset of λ-expressions, $\bigcup X$ means $\bigcup \mathcal{J}(X)$.

Theorem 5.1. *The β-rule of conversion is valid in $\mathcal{J}$, i.e., if $M \xrightarrow{*} M'$, then $M \equiv M'$.*

Proof. Since $M \xrightarrow{*} M'$ we have $\mathcal{A}(M') \subset \mathcal{A}(M)$, and then $M' \subseteq M$. Now suppose $M \xrightarrow{*} N$, then as $M \xrightarrow{*} M'$ we know, by the Church–Rosser Theorem, that there exists an N' such that $N \xrightarrow{*} N'$ and $M' \rightarrow N'$, and hence $\phi(N) < \phi(N')$. So we have $M \subseteq M'$. □

We extend the pure λ-calculus by adding a constant Ω and closing under abstraction and application, and we let Λ be the set of terms of the extended language. We consider not only β-reductions but also an ω-rule of conversion defined by replacing any subexpression of the form ΩM by Ω. We write this kind of reduction $N \underset{\omega}{\xrightarrow{*}} N'$. Let $\mathscr{C}[\,]$ denote any context (see [9]), i.e., a λ-Ω expression with one subexpression missing, and let $\mathscr{C}[M]$ be the corresponding expression where M stands in the position of the previously missing subexpression.

Proposition 5.2. *$\mathscr{C}[\Omega] \subseteq \mathscr{C}[M]$ for any context M and any expression M.*

Proof. The set of expressions reducible from $\mathscr{C}[\Omega]$ is isomorphic to a subset of the one reducible from $\mathscr{C}[M]$. Moreover, $\phi(N[x\backslash\Omega]) < \phi(N[x\backslash M])$ for any λ-Ω expressions M, N. Hence, $\mathscr{C}[\Omega] \subseteq \mathscr{C}[M]$ by definition of $\subseteq$. □

Theorem 5.3. *The ω-rule of conversion is valid in $\mathcal{J}$, i.e., if $M \underset{\omega}{\xrightarrow{*}} M'$, then $M \equiv M'$.*

Proof. By the above proposition, we already know that $M' \subseteq M$. Now suppose $M \xrightarrow{*} N$; we can easily show (by an induction on the pair $\langle l, l' \rangle$, if l and l' are the length of the reductions $M \xrightarrow{*} N$ and $M \underset{\omega}{\xrightarrow{*}} M'$) the existence of an N' such that $N \underset{\omega}{\xrightarrow{*}} N'$ and $M' \xrightarrow{*} N'$. Hence $\phi(N) = \phi(N')$ and then $M \subseteq M'$. □

We turn now to the main point of this paper, i.e., we show that for any context $\mathscr{C}[\,]$ we have $\mathscr{C}[M]\subseteq\mathscr{C}[M']$ if $M\subseteq M'$. So, using the definition of $\subseteq$, we need to show that if $\mathscr{C}[M]\xrightarrow{*}N$, then there is an N' such that $\mathscr{C}[M']\xrightarrow{*}N'$ and $\phi(N)<\phi(N')$. But all we know is that for any approximation of M we can have a better one for M'. Therefore, in order to compare $\mathscr{C}[M]$ and $\mathscr{C}[M']$ we try to point out the approximation of M needed by any reduction from $\mathscr{C}[M]$ to some N. That is Welch's conjecture about inside-out reductions ([15]), which we prove later. Using Welch's notations, let $\mathscr{C}[M]\xrightarrow[M]{*}N$ designate any reduction

$$\mathscr{C}[M]=M_0\xrightarrow{R_1}M_1\xrightarrow{R_2}M_2\xrightarrow{R_3}\cdots\xrightarrow{R_n}M_n=N,$$

where, for all i $(1\leqslant i\leqslant n)$, the redex R_i contracted between M_{i-1} and M_i is not a residual of a redex internal to the subexpression M in M_0. Similarly, if $\mathscr{F}$ is a set of redexes in M_0, we write $M_0\xrightarrow[\mathscr{F}]{*}M_n$ if none of the R_i is a residual of a redex of $\mathscr{F}$. Hence, if $\mathscr{F}$ is the set of all redexes of M, we have $\mathscr{C}[M]\xrightarrow[M]{*}N$ iff $\mathscr{C}[M]\xrightarrow[\mathscr{F}]{*}N$. Moreover, let $M[\mathscr{F}\backslash\Omega]$ denote the substitution in M of all redexes of $\mathscr{F}$ by the constant Ω. Now we can state what we want. (The proof is postponed until the next section).

Proposition 5.4. *For any context $\mathscr{C}[\,]$ and any expression M, if $\mathscr{C}[M]\xrightarrow{*}N$, then there are expressions M' and N' such that $M\xrightarrow{*}M'$, $N\xrightarrow{*}N'$, and $\mathscr{C}[M']\xrightarrow[M']{*}N'$.*

Lemma 5.5. *Given a set of redexes $\mathscr{F}$ in an expression M, if $M\xrightarrow[\mathscr{F}]{*}M'$ and if $\mathscr{F}'$ is the set of residuals of the redexes of $\mathscr{F}$ in M', then $M[\mathscr{F}\backslash\Omega]\xrightarrow{*}M'[\mathscr{F}'\backslash\Omega]$.*

Proof. Follows by induction on the length of the reduction from M to M', which is of the form $M\xrightarrow{R}M_1\xrightarrow[\mathscr{F}_1]{*}M'$, where the redex R, first contracted, is not one of the redexes of $\mathscr{F}$ and $\mathscr{F}_1$ is the set of residuals of the redexes of $\mathscr{F}$ in M_1. Then, depending on the relative position of R and $\mathscr{F}$, we obviously have $M[\mathscr{F}\backslash\Omega]\xrightarrow{*}M_1[\mathscr{F}_1\backslash\Omega]$. □

Lemma 5.6. *If a, b are in the set $\mathscr{N}$ of ω-β normal forms and such that $a<b$, then $\mathscr{C}[a]\subseteq\mathscr{C}[b]$ for any context $\mathscr{C}[\,]$.*

Proof. The lemma is a corollary of Proposition 5.2, since if $a<b$, then a matches b except for some Ω's □

Lemma 5.7. *For any context* $\mathscr{C}[\,]$ *and expression* M

$$\mathscr{C}[M] \equiv \bigcup \{\mathscr{C}[a]:\ a \in \mathscr{A}(M)\}.$$

Proof. Let $X = \{\mathscr{C}[a]:\ a \in \mathscr{A}(M)\}$. By Lemma 5.6, as $\mathscr{A}(M)$ is a directed set, X is also directed and $\bigcup X$ exists since $\hat{\mathscr{N}}$ is complete for directed subsets.

First, if a is in $\mathscr{A}(M)$ there is an expression N such that $M \xrightarrow{*} N$ and $a = \phi(N)$. So we have, by Proposition 5.2 $\mathscr{C}[a] \subseteq \mathscr{C}[N]$ since a matches N except in some Ω's. But as $M \xrightarrow{*} N$ we have, too, $\mathscr{C}[M] \xrightarrow{*} \mathscr{C}[N]$ and $\mathscr{C}[M] \equiv \mathscr{C}[N]$ by Theorem 5.1. Hence, we get $\mathscr{C}[a] \subseteq \mathscr{C}[M]$ for any a in $\mathscr{A}(M)$. Therefore $\mathscr{C}[M]$ is an upper bound of X and $\bigcup X \subseteq \mathscr{C}[M]$.

Conversely, if $\mathscr{C}[M] \xrightarrow{*} N$, there are M' and N' such that $M \xrightarrow{*} M'$, $N \xrightarrow{*} N'$, and $\mathscr{C}[M'] \xrightarrow[M']{*} N'$ (by Proposition 5.4). Let $\mathscr{F}$ be the set of all redexes of M'; we have $\mathscr{C}[M'] \xrightarrow[\mathscr{F}]{*} N'$. By Lemma 5.5, if $\mathscr{F}'$ is the set of the residuals of the redexes of $\mathscr{F}$ in N', we have $\mathscr{C}[M'][\mathscr{F}\backslash\Omega] \xrightarrow{*} N'[\mathscr{F}'\backslash\Omega]$. Moreover, $\mathscr{C}[M'][\mathscr{F}\backslash\Omega] \xrightarrow[\omega]{*} \mathscr{C}[\phi(M')]$ and since ω and β-conversions are valid we get

$$\phi(N) < \phi(N') = \phi(N'[\mathscr{F}'\backslash\Omega]) \subseteq N'[\mathscr{F}'\backslash\Omega] \equiv \mathscr{C}[M'][\mathscr{F}\backslash\Omega] \equiv \mathscr{C}[\phi(M')].$$

Then for any N such that $\mathscr{C}[M] \xrightarrow{*} N$ there is an M' such that $M \xrightarrow{*} M'$ and $\phi(N) \subseteq \mathscr{C}[\phi(M')]$. Since $\mathscr{C}[M] \equiv \bigcup \{a:\ a \in \mathscr{A}(\mathscr{C}[M])\}$, we have $\mathscr{C}[M] \subseteq \bigcup X$. □

Theorem 5.8. *If* $M \subseteq M'$, *then* $\mathscr{C}[M] \subseteq \mathscr{C}[M']$ *for any context* $\mathscr{C}[\,]$.

Proof. Since $M \subseteq M'$, for any a in $\mathscr{A}(M)$ there is a b in $\mathscr{A}(M')$ such that $a < b$. Hence, by Lemma 5.6, $\mathscr{C}[a] \subseteq \mathscr{C}[b]$. Since $\hat{\mathscr{N}}$ is complete, we get

$$\bigcup \{\mathscr{C}[a]:\ a \in \mathscr{A}(M)\} \subseteq \bigcup \{\mathscr{C}[b]:\ b \in \mathscr{A}(M')\}$$

ınd, by Lemma 5.7, $\mathscr{C}[M] \subseteq \mathscr{C}[M']$. □

6. Inside-out reductions

In order to show Proposition 5.4, we follow Welch [15] and define an inside-out reduction as any reduction:

$$M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \xrightarrow{R_3} \cdots \xrightarrow{R_n} M_n = M'$$

where, for all i, j such that $i < j$ and $1 \leqslant i \leqslant n-1$, $2 \leqslant j \leqslant n$, the redex R_j (contracted between M_{j-1} and M_j) is not a residual of a redex internal to R_i in M_{i-1}. Let $M \xrightarrow[\text{i.o.}]{*} M'$ be a notation for any such inside-out reduction from M to M'. Welch conjectured the completeness of inside-out reductions, i.e., if $M \xrightarrow{*} N$, then there exists N' such that $N \xrightarrow{*} N'$ and $M \xrightarrow[\text{i.o.}]{} N'$. So, as pointed out by Welch:

Proposition 6.1. *If inside-out reductions are complete, then Proposition 9 is true.*

Proof. Let $\mathscr{C}[M] \xrightarrow{*} N_1$, there is N such that $N_1 \xrightarrow{*} N$ and $\mathscr{C}[M] \xrightarrow[\text{i.o.}]{*} N$, as inside-out reductions are complete. Let us prove, by induction on the size of the context $\mathscr{C}[]$, that if $\mathscr{C}[M] \xrightarrow[\text{i.o.}]{*} N$, there is an M' such that $M \to M'$ and $\mathscr{C}[M'] \xrightarrow[M']{*} N$.

1) If $\mathscr{C}[] = []$, then $\mathscr{C}[M] = M$ and $M' = \mathscr{C}[M'] = N$.
2) If $\mathscr{C}[] = \lambda x \cdot \mathscr{C}_1[]$, the induction works easily.
3) If $\mathscr{C}[] = M_1\mathscr{C}_1[]$, as the reduction $\mathscr{C}[M] \xrightarrow[\text{i.o.}]{*} N$ is inside-out, we have $M_1 \xrightarrow[\text{i.o.}]{*} M_1'$, $\mathscr{C}_1[M] \xrightarrow[\text{i.o.}]{*} N'$ and $M_1'N' \xrightarrow[N']{*} N$. Hence, by induction there is an M' such that $M \xrightarrow{*} M'$ and $\mathscr{C}_1[M'] \xrightarrow[M']{*} N'$. Thus $M \xrightarrow{|*|} M'$ and $\mathscr{C}[M'] = M_1\mathscr{C}_1[M'] \xrightarrow[M']{*} N$.
4) If $\mathscr{C}[] = \mathscr{C}_1[]M_1$, we have the same proof. □

7. A labelled λ-calculus

The problem is now to keep track of the redexes contracted in some reduction $M \xrightarrow{*} N$ in order to be able to reorder them in an inside-out way, and to show the inside-out completeness. We do this by introducing a new set of λ-expressions (Λ') defined on a set of labels $\mathscr{L}$ as follows. Let $\mathscr{L}_0 = \{a, b, c \cdots\}$ be an infinite set of letters. We consider the set $\mathscr{L}$ of all strings of characters formed on $\mathscr{L}_0$, with any level of overlining and underlining. So expressions of $\mathscr{L}$ are:

$$\begin{cases} a & \text{if } a \in \mathscr{L}_0 \\ \alpha\beta & \text{if } \alpha, \beta \in \mathscr{L} \\ \bar{\alpha} & \text{if } \alpha \in \mathscr{L} \\ \underline{\alpha} & \text{if } \alpha \in \mathscr{L} \end{cases}$$

and expressions of Λ' are:

$$\begin{cases} x^\alpha & \text{if } \alpha \in \mathscr{L} \text{ and } x \in V \\ (\lambda x \cdot M)^\alpha & \text{if } \alpha \in \mathscr{L} \text{ and } M \in \Lambda' \\ (MN)^\alpha & \text{if } \alpha \in \mathscr{L} \text{ and } M, N \in \Lambda' \end{cases}$$

Thus, the labelled λ-expressions are like usual λ-expressions except that every subexpression has an arbitrary label. This λ-calculus is a generalization of one of

Wadsworth since, instead of considering integers as exponents, we have strings o characters. For any label α and expression M of Λ', we define $\alpha\cdot M$ as:

$$\begin{cases} \alpha\cdot x^{\beta} = x^{\alpha\beta} \\ \alpha\cdot(\lambda x\cdot M)^{\beta} = (\lambda x\cdot M)^{\alpha\beta} \\ \alpha\cdot(MN)^{\beta} = (MN)^{\alpha\beta} \end{cases}$$

and the substitution operation is defined by:

$$\begin{cases} x^{\alpha}[x\backslash N] = \alpha_0 N \\ y^{\alpha}[x\backslash N] = y^{\alpha} \\ (\lambda y\cdot M)^{\alpha}[x\backslash N] = (\lambda y\cdot M[x\backslash N])^{\alpha} \\ (MM')^{\alpha}[x\backslash N] = (M[x\backslash N]M'[x\backslash N])^{\alpha} \end{cases}$$

where we forget the difficulties due to α-conversion. Then the β-rule is defined (by monotony) from:

$$((\lambda x\cdot M)^{\alpha}N)^{\beta} \to \beta\bar{\alpha}\cdot M[x\backslash \underline{\alpha}\cdot N]$$

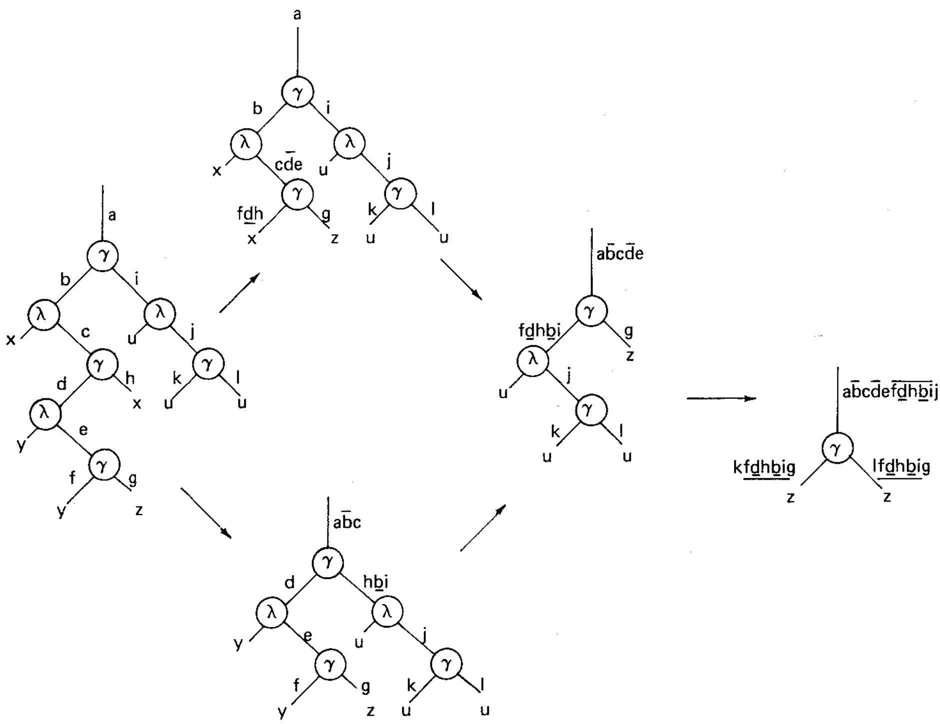

Fig. 1. Reductions from $((\lambda x\cdot(\lambda y\cdot(y^f z^g)^e)^d x^h)^c)^b(\lambda u\cdot(u^k u^l)^j)^i)^a$

(We do not care for the precedences between the $\cdot$ and substitution operators because they commute). Furthermore we will allow this reduction iff some predicate $\mathscr{P}(\alpha, \beta)$ is verified. So, for instance, using a graph notation for λ-expressions (see Morris [9] where nodes λ and γ corresponds to abstraction and application), we have figure 1 if we suppose $\mathscr{P}(\alpha, \beta)$ always true. In fact, we can restrict our attention to λ-expressions labelled by a set $\mathscr{L}'$ of labels defined as containing:

$$\begin{cases} a & \text{if } a \in \mathscr{L}_0 \\ \alpha\bar{\beta}\gamma & \text{if } \alpha, \beta, \gamma \in \mathscr{L}' \\ \alpha\underline{\beta}\gamma & \text{if } \alpha, \beta, \gamma \in \mathscr{L}' \end{cases}$$

and it is clear that expressions labelled by $\mathscr{L}'$ keep their labels in $\mathscr{L}'$ after some β-reductions. We remark too that other λ-calculus languages are obtainable from this one by some homomorphism: for instance Wadsworth's typed λ-calculus and Morris' definition of descendants. Let the height $h(\alpha)$ of a label α of $\mathscr{L}$ be defined by:

$$\begin{cases} h(a) = o & \text{if } a \in \mathscr{L}_0 \\ h(\alpha\beta) = \lceil h(\alpha), h(\beta) \rceil & \text{if } \alpha, \beta \in \mathscr{L} \\ h(\underline{\alpha}) = 1 + h(\alpha) & \text{if } \alpha \in \mathscr{L}. \end{cases}$$

and let the degree of a redex be the label of its abstraction part. Hence, we have degree $(((\lambda x \cdot M)^\alpha N)^\beta) = \alpha$.

Proposition 7.1. *The residuals of a redex R have the same degree as R.*

Proof. Suppose $M \xrightarrow{R} N$ and R is a redex, in M, of the form $R = ((\lambda x \cdot P)^\alpha Q)^\beta$. If $S = ((\lambda y \cdot T)^\gamma U)^\delta$ is another redex of M, we show by cases that residual(s) of S in N have the same degree γ than S in M.

1) If R and S are 2 disjoint expressions, it is obvious.

2) If S is in R, then S is in P or Q and the contraction of R may have only an effect on the external label δ of S.

3) If R is in S, then R is in T or U and the contraction of R has no effect on the degree γ of S. □

Proposition 7.2. *If $\mathscr{P}(\alpha, \beta)$ implies $\mathscr{P}(\alpha, \gamma\beta)$ for any labels α, β, γ of $\mathscr{L}$, then the β-rule of the labelled calculus is Church–Rosser.*

Proposition 7.3. *If*

1) *$\mathscr{P}(\alpha, \beta)$ implies $\mathscr{P}(\alpha, \gamma\beta)$ for any labels α, β, γ of $\mathscr{L}$*

2) *the set $\{h(\alpha) \mid \mathscr{P}(\alpha, \beta)$ is true$\}$ is bounded then any labelled λ-expression strongly normalizes (i.e., any reduction in this labelled calculus has a finite length).*

The proofs of these Propositions are given in the appendix, by the usual techniques. We go back to the inside-out completeness and we will use letters as M, N to designate expressions of Λ and U, V for labelled λ-expressions of Λ'.

Theorem 7.4. *If* $M \xrightarrow{*} N$, *then there is an expression* N' *such that* $M \xrightarrow[\text{i.o.}]{*} N'$ *and* $N \xrightarrow{*} N'$.

Proof. Let U be the labelled λ-expression obtained from M by labelling all the subexpressions of M with a different letter of $\mathscr{L}_0$. We can associate to the reduction $M \xrightarrow{*} N$, an isomorphic labelled reduction $U \xrightarrow{*} V$. More precisely, this reduction can be written:

$$U = U_0 \xrightarrow{R_1} U_1 \xrightarrow{R_2} U_2 \xrightarrow{R_3} \cdots \xrightarrow{R_n} U_n = V$$

Let us now consider the predicate $\mathscr{P}(\alpha, \beta)$ defined on labels by:

$$\mathscr{P}(\alpha, \beta) \text{ is true iff } \alpha = \text{degree } (R_i) \text{ for some } i \ (1 \leqslant i \leqslant n)$$

The two assumptions of Proposition 3′ are verified and, hence, U strongly normalizes. Let V' be the normal form of U, then $V \to V'$, because the Church–Rosser assumption is true and for instance, any innermost reduction reaches the normal form V'. Let

$$U = V_0 \xrightarrow{S_1} V_1 \xrightarrow{S_2} V_2 \xrightarrow{S_3} \cdots \xrightarrow{S_m} V_m = V'$$

be such an innermost reduction. (We then have, for all i, degree $(S_i) =$ degree (R_j) for some j between 1 and n). We claim that this reduction is inside-out. Suppose $i < j$ for some i, j between 1 and m and suppose S_j is a residual of a redex S'_j internal to S_i in V_{i-1}. By Proposition 1′, we have degree $(S'_j) =$ degree (S_j) and then, as the predicate $\mathscr{P}$ is true for S_j, $\mathscr{P}$ is also true for S'_j. The reduction from U to V' is thus not an innermost reduction and we have a contradiction. Let N' be the λ-expression obtained by erasing the labels of V'. As an isomorphic reduction of Λ corresponds to any labelled reduction, we have $N \xrightarrow{*} N'$ and $M \xrightarrow[\text{i.o.}]{*} N'$. □

In fact, with the same method, if $M \xrightarrow{*} M'$ and $M \xrightarrow{*} M''$, we have an N such that $M' \xrightarrow[\text{i.o.}]{*} N$ and $M'' \xrightarrow[\text{i.o.}]{*} N$. In the above proof, we do not need the strong normalization property, but only the normalization of innermost reductions which is easier to prove. But the strong normalization property shows that we can extract arbitrarily large finite Church–Rosser subsets in the set of all reductions of a given expression. The inside-out way is just a particular order in such a subset.

Conclusion

The interpretation $\langle \mathscr{I}, \hat{\mathscr{N}} \rangle$, although strongly inspired by Scott's theory of computation, is purely algebraic. Here, we do not have a definition of application as in Scott [11, 12] or Welch [16]. But with the help of labelled calculus, any expression can be considered as the limit of expressions having a normal form. If we think of λ-expressions as programs, the interpretation $\langle \mathscr{I}, \hat{\mathscr{N}} \rangle$ seems to be the minimal one to consider. Thus we expect that $\langle \mathscr{I}, \hat{\mathscr{N}} \rangle$ is some kind of free interpretation. This interpretation is weaker than the one of Hyland [4] and Welch [16] because we do not identify the expressions Ω and $\lambda x \cdot \Omega$. This choice has to be justified, maybe by adding constants and new conversion rules to them, but it is clear that we could have equated Ω and $\lambda x \cdot \Omega$ in the whole paper with minor modifications of the different proofs (see [6]). We remark too that there exist two different ways of proving the congruence induced by the interpretation: 1) using the completeness of the inside-out reductions as suggested by Welch [15]; 2) using a characterization à la Böhm of the order defined by the interpretation as done by Hyland [4]. Here we use the first method and we show a "strong" completeness of the inside-out reductions; but it would have been sufficient for the congruence to satisfy the following "weak" property: for any expressions M, N such that $M \xrightarrow{*} N$, there is an expression N' such that $M \underset{\text{i.o.}}{\xrightarrow{*}} N'$ and $\phi(N) < \phi(N')$ (see Wadsworth [17]). Another question is to take into account extensionality and build an algebraic interpretation where the η-rule is valid. This is done by Hyland [4]. Finally, the labelled λ-calculus seems interesting in itself [7], since we can capture the history of any reduction in the labels.

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Appendix 1.

The Church–Rosser property in the labelled calculus (by the Taït-Martin Löf method).

Let $\mathscr{C}(M)$ be defined by:

$$\begin{cases} \mathscr{C}(x^\alpha) & = x^\alpha \\ \mathscr{C}((\lambda x \cdot M)^\alpha) & = (\lambda x \cdot \mathscr{C}(M))^\alpha \\ \mathscr{C}(((\lambda x \cdot M)^\alpha N)^\beta) & = \beta\bar{\alpha} \cdot \mathscr{C}(M)[x \backslash \underline{\alpha} \cdot \mathscr{C}(N)] \quad \text{if } \mathscr{P}(\alpha, \beta) \\ \mathscr{C}((MN)^\alpha) & = (\mathscr{C}(M)\mathscr{C}(N))^\alpha \quad \text{otherwise} \end{cases}$$

Let $M \rightarrow M'$ denote a parallel step of reduction and be defined by the following inference rules and axiom:

$$\text{-I-} \qquad x^\alpha \rightarrow x^\alpha$$

$$\text{-II-} \qquad \frac{M \rightarrow M'}{(\lambda x \cdot M)^\alpha \rightarrow (\lambda x \cdot M')^\alpha}$$

$$\text{-III-} \qquad \frac{M \rightarrow M', N \rightarrow N'}{(MN)^\alpha \rightarrow (M'N')^\alpha}$$

$$\text{-IV-} \qquad \frac{M \rightarrow M', N \rightarrow N'}{((\lambda x \cdot M)^\alpha N)^\beta \rightarrow \beta\bar{\alpha} \cdot M'[x \backslash \underline{\alpha} \cdot N']} \qquad \text{if } \mathscr{P}(\alpha, \beta)$$

Morever, we suppose $\mathscr{P}(\alpha, \beta)$ implies $\mathscr{P}(\alpha, \gamma\beta)$ for any labels α, β, γ.

First, we notice that the associativity of concatenation implies:

$$\alpha \cdot (\beta \cdot M) = \alpha\beta \cdot M$$

Lemma 1. $\alpha \cdot (M[x\backslash N]) = (\alpha \cdot M)[x\backslash N]$

Proof. By cases on M. The only problem is when $M = x^\beta$. Then the associativity of concatenation gives the answer. □

Lemma 2. *If $x \neq y$ and x is not free in N', then*:

$$M[x\backslash N][y\backslash N'] = M[y\backslash N'][x\backslash N[y\backslash N']]$$

Proof. By induction on the size of M. The only problem is when $M = x^\alpha$. Then we apply Lemma 1. □

Lemma 3. *If $M \to M'$, then $\alpha \cdot M \to \alpha \cdot M'$*

Proof. By cases on the rule or axiom used for $M \to M'$. The only interesting case is when:

$$M = ((\lambda x \cdot M_1)^\gamma M_2)^\beta \to M' = \beta\bar{\gamma} \cdot M_1'[x\backslash\gamma \cdot M_2']$$

with $M_1 \to M_1'$, $M_2 \to M_2'$ and $\mathcal{P}(\gamma, \beta)$. Then:

$$\alpha \cdot M = ((\lambda x \cdot M_1)^\gamma M_2)^{\alpha\beta} \quad \text{and} \quad \alpha \cdot M' = \alpha \cdot (\beta\bar{\gamma} \cdot M_1'[x\backslash\gamma \cdot M_2'])$$

As $\mathcal{P}(\gamma, \beta)$ implies $\mathcal{P}(\gamma, \alpha\beta)$, we have by rule IV:

$$\alpha \cdot M = ((\lambda x \cdot M_1)^\gamma M_2)^{\alpha\beta} \to \alpha\beta\bar{\gamma} \cdot M_1'[x\backslash\gamma \cdot M_2']$$

and by the associativity of concatenation $\alpha \cdot M \to \alpha \cdot M'$. □

Lemma 4. *If $M \to M'$ and $N \to N'$, then $M[x\backslash N] \to M'[x\backslash N']$*

Proof. By induction of the size of M.

1) M is a variable:

a) $M = x^\alpha = M'$. Then we use Lemma 3.

b) $M = y^\alpha = M'$. Then obvious by axiom I.

2) M is not a variable and we have several cases according to the rule used for $M \to M'$. The only interesting one is when:

$$M = ((\lambda y \cdot M_1)^\alpha M_2)^\beta \to M' = \beta\bar{\alpha} \cdot M_1'[y\backslash\underline{\alpha} \cdot M_2']$$

with $M_1 \to M_1'$, $M_2 \to M_2'$ and $\mathcal{P}(\alpha, \beta)$. Then, ignoring α-conversions, we have:

$$M[x\backslash N] = ((\lambda y \cdot M_1[x\backslash N])^\alpha M_2[x\backslash N])^\beta$$

$$\begin{aligned} M'[x\backslash N'] &= (\beta\bar{\alpha} \cdot M_1'[y\backslash\underline{\alpha} \cdot M_2'])[x\backslash N']) \\ &= \beta\bar{\alpha} \cdot (M_1'[y\backslash\underline{\alpha} \cdot M_2'][x\backslash N'] && \text{(by Lemma 1)} \\ &= \beta\bar{\alpha} \cdot (M_1'[x\backslash N'][y\backslash(\underline{\alpha} \cdot M_2')[x\backslash N']]) && \text{(by Lemma 2)} \\ &= \beta\bar{\alpha} \cdot (M_1'[x\backslash N'][y\backslash\underline{\alpha} \cdot (M_2'[x\backslash N'])]) && \text{(by Lemma 1)} \end{aligned}$$

By induction, we know that $M_1[x\backslash N] \to M_1'[x\backslash N']$ and $M_2[x\backslash N] \to M_2'[x\backslash N']$, and using rule IV, we have $M[x\backslash N] \to M'[x\backslash N']$. □

Lemma 5. *If $M \to M'$, then $M' \to \mathscr{C}(M)$.*

Proof. By induction on the size of M. There are two interesting cases:
1) When:

$$M = ((\lambda x \cdot M_1)^\alpha M_2)^\beta \to M' = (M_3' M_2')^\beta$$

with $(\lambda x \cdot M_1)^\alpha \to M_3'$, $M_2 \to M_2'$ and $\mathscr{P}(\alpha, \beta)$. Then it is clear that $M_3' = (\lambda x \cdot M_1')^\alpha$ and $M_1 \to M_1'$. Hence we have by induction $M_1' \to \mathscr{C}(M_1)$ and $M_2' \to \mathscr{C}(M_2)$. Then, using rule IV:

$$M' = ((\lambda x \cdot M_1')^\alpha M_2')^\beta \to \beta\bar{\alpha} \cdot \mathscr{C}(M_1)[x\backslash \underline{\alpha} \cdot \mathscr{C}(M_2)] = \mathscr{C}(M)$$

2) When:

$$M = ((\lambda x \cdot M_1)^\alpha M_2)^\beta \to M' = \beta\bar{\alpha} \cdot M_1'[x\backslash \underline{\alpha} \cdot M_2']$$

with $M_1 \to M_1'$, $M_2 \to M_2'$ and $\mathscr{P}(\alpha, \beta)$. Then we have by induction $M_1' \to \mathscr{C}(M_1)$ and $M_2' \to \mathscr{C}(M_2)$. Hence, by Lemma 3: $\underline{\alpha} \cdot M_2' \to \underline{\alpha} \cdot \mathscr{C}(M_2)$, and by Lemma 4 and 3:

$$M' \to \beta\bar{\alpha} \cdot \mathscr{C}(M_1)[x\backslash \underline{\alpha} \cdot \mathscr{C}(M_2)] = \mathscr{C}(M). \ \square$$

Lemma 6. *If $M \to M'$ and $M \to M''$, then there is an N such that $M' \to N$ and $M'' \to N$.*

Proof. We take $N = \mathscr{C}(M)$ and use Lemma 5. □

Proposition. *If $M \xrightarrow{*} M'$ and $M \xrightarrow{*} M''$, then there is an N such that $M' \xrightarrow{*} N$ and $M'' \xrightarrow{*} N$.*

Proof. By induction on the sum of length of the reductions $M \xrightarrow{*} M'$ and $M \xrightarrow{*} M''$. □

Appendix 2.

Strong normalization in the labelled λ-calculus by a method due to D. van Daalen. We suppose:
(1) $\mathscr{P}(\alpha, \beta)$ implies $\mathscr{P}(\alpha, \gamma\beta)$
(2) $\{h(\alpha) \mid \mathscr{P}(\alpha, \beta) \text{ is true}\}$ is bounded
Hence, we have the Church–Rosser property. Let us write $\tau(N)$ for the external label of N. So $\tau(x^\alpha) = \tau((\lambda x \cdot M)^\alpha) = \tau((MN)^\alpha) = \alpha$ and we call $\mathscr{SN}$ the set of strongly normalizable labelled λ-expressions.

Lemma 1. *If $(\cdots((MN_1)^{\beta_1}N_2)^{\beta_2}\cdots N_n)^{\beta_n} \xrightarrow{*} (\lambda x \cdot N)^\alpha$, then we have $h(\tau(M)) \leq h(\alpha)$.*

Proof. By induction on n. If $n = 0$, this is clearly true. Otherwise, we must have:

$$(\cdots((MN_1)^{\beta_1}N_2)^{\beta_2}\cdots N_{n-1})^{\beta_{n-1}} \xrightarrow{*} (\lambda y \cdot P)^{\gamma}$$

and:

$$((\lambda y \cdot P)^{\gamma}N_n)^{\beta_n} \to \beta_n\bar{\gamma} \cdot P[y\backslash\underline{\gamma} \cdot N_n] \xrightarrow{*} (\lambda x \cdot N)^{\alpha}$$

Hence, we get $h(\tau(M)) \leq h(\gamma)$ by induction. We also have:

$$h(\gamma) < h(\bar{\gamma}) \leq h(\tau(\beta_n\bar{\gamma} \cdot P[y\backslash\gamma \cdot N_n])) \leq h(\alpha). \ \square$$

Lemma 2. *If $M[x\backslash N] \xrightarrow{*} (\lambda y \cdot P)^{\alpha}$, we only have two cases:*

1) $M \xrightarrow{*} (\lambda y \cdot M')^{\alpha}$ *and* $M'[x\backslash N] \xrightarrow{*} P$

or

2) $M \xrightarrow{*} M' = (\cdots((x^{\beta}M_1)^{\beta_1}M_2)^{\beta_2}\cdots M_n)^{\beta_n}$ *and* $M'[x\backslash N] \xrightarrow{*} (\lambda y \cdot P)^{\alpha}$

Proof. Application of Propositions 2.1 and 2.2. $\square$

Lemma 3. *If M, N are in $\mathcal{SN}$, then $M[x\backslash N]$ is in $\mathcal{SN}$.*

Proof. Let m be the upper bound of the set $\{h(\alpha) \mid \mathcal{P}(\alpha, \beta) \text{ is true}\}$, which exists by assumption (2), and prof (M) be the maximal length of reductions starting from M in $\mathcal{SN}$. We do an induction on the triple:

$$\langle h(\tau(N)) - m, \mathrm{prof}(M), \|M\|\rangle$$

where $\|M\|$ is the size of M.

The only interesting case is when:

$$M = (M_1M_2)^{\alpha} \text{ and } M_1[x\backslash N] \xrightarrow{*} (\lambda y \cdot P_1)^{\alpha_1}$$

We know by induction that $M_1[x\backslash N]$ and $M_2[x\backslash N]$ are in $\mathcal{SN}$, but we wonder if $((\lambda y \cdot P_1)^{\alpha_1}M_2[x\backslash N])^{\alpha}$ is in $\mathcal{SN}$, i.e., if $M' = \alpha\overline{\alpha_1} \cdot P_1[y\backslash\underline{\alpha_1} \cdot M_2[x\backslash N]]$ is in $\mathcal{SN}$. Then Lemma 2 tells us there are two subcases:

1) $M_1 \xrightarrow{*} (\lambda y \cdot M_1')^{\alpha_1}$ and $M_1'[x\backslash N] \xrightarrow{*} P_1$. Then by Lemmas 2, 3, 4 of the Church–Rosser proof, we get:

$$\alpha\overline{\alpha_1} \cdot M_1'[y\backslash\underline{\alpha_1} \cdot M_2][x\backslash N] = \alpha\overline{\alpha_1} \cdot M_1'[x\backslash N][y\backslash\underline{\alpha_1} \cdot M_2[x\backslash N]]$$

$$\xrightarrow{*} \alpha\overline{\alpha_1} \cdot P_1[y\backslash\underline{\alpha_1} \cdot M_2[x\backslash N]] = M'$$

But $M = (M_1M_2)^{\alpha} \xrightarrow{*} ((\lambda y \cdot M_1')^{\alpha_1}M_2)^{\alpha} \to \alpha\overline{\alpha_1} \cdot M_1'[x\backslash\underline{\alpha_1} \cdot M_2]$. Hence:

$$\operatorname{prof}(\alpha\overline{\alpha_1}\cdot M_1'[x\backslash \underline{\alpha}_1\cdot M_2])<\operatorname{prof}(M)$$

and by induction M' is in $\mathcal{SN}$.

2) $M_1 \xrightarrow{*} Q_1 = (\cdots((x^{\beta}N_1)^{\beta_1}N_2)^{\beta_2}\cdots N_n)^{\beta_n}$ and $Q_1[x\backslash N] \xrightarrow{*} (\lambda y\cdot P_1)^{\alpha_1}$. As $M_1[x\backslash N]$ is in $\mathcal{SN}$ and $M_1[x\backslash N] \xrightarrow{*} Q_1[x\backslash N]$, we have P_1 in $\mathcal{SN}$. Moreover:

$$Q_1[x\backslash N]=(\cdots(((\beta\cdot N)N_1')^{\beta_1}N_2'))^{\beta_2}\cdots N_n')^{\beta_n}$$

where $N_i'=N_i[x\backslash N]$ for all i. Hence, using Lemma 1:

$$h(\tau(N))\leqslant h(\tau(\beta\cdot N))\leqslant h(\alpha_1)<h(\underline{\alpha}_1)\leqslant h(\tau(\underline{\alpha}_1\cdot M_2[x\backslash N]))$$

and by induction M' is in $\mathcal{SN}$. □

Proposition. *If $\mathcal{P}$ verifies assumptions* (1) *and* (2), *every expression M strongly normalizes.*

Proof. By induction on the size of M and application of Lemma 3. □

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